NASA TECHNICAL MEMORANDUM NASA TM-75466

POSSIBILITIES AND LIMITATIONS OF ROD-BEAM THEORIES

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(NASA-TM-75466) POSSIBILITIES AND LIMITATIONS OF ROD-BEAM THEORIES (National Aeronautics and Space Administration) 38 p HC A03/MF A01 CSCL 20K **N79-28613**

Unclas

G3/39 31661

Translation of "Möglichkeiten und Grenzen der Stabtheorien," Deutsche Forschungs- und Versuchsanstalt für Luft- und Raumfahrt, Forschungsbericht, Institut für Strukturmechanik, Braunschweig, West Germany. DFVLR-FB 78-13, 1978, pp 1-41

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. JULY 1979

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D. Petersen[1]

1. Basic Problems with Rod-Beam Theories /7*

In rod-beam theory the outstanding quantity of the length of an actual spatial continuum is used to simplify static and dynamic analysis. Thus the description of a three-dimensional condition is impossible in one dimension without hypotheses such as Bernouilli's, which suggests flat rod cross-sections during deformation. Wlassow [12] gives a detailed description of the basic assumptions in rod theory. Such hypotheses also have effects on the stress and distortion tensors.

It is unusual to speak of a stress or distortion tensor in rods, since rod-beam theories are based upon the assumption that only a few components of these tensors are other than zero. When dealing with distortions, one confines oneself essentially to the definition of a strain along the longitudinal axis of the rod. Because of bending and curving, the strain varies across the cross section.

Rod-beam theory requires that no normal stresses should arise transverse to the rod axis. Thus stress-free transverse strains must be allowed, but they alter the cross-section dimensions only infinitesimally. Therefore the contour of the cross-section is maintained in the deformed state.

As a matter of principle, shearing deformation is regarded

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* Numbers in the margin indicate pagination in the foreign text.

as negligible in rods, as required by the hypothesis of flat cross sections in the deformed state. Shearing stresses per se are determined from the so-called stress functions, equilibrium equations with expressions for the distribution of such stresses over the cross section. If in special cases the displacement of the rod due to shearing is not negligible, an approximation is used in which such displacements are superimposed upon the displacements due to bending. At the same time, however, the effects of shearing on the distortions are regarded as negligible. These approximations and equivalent procedures in considerations of secondary shearing due to curving force torsion are discussed elsewhere [4,5,11].

It is obvious that such approximations are valid only under particular circumstances. This work is thus limited to long rods in which both shearing distortions and displacement due to shearing remain negligible.

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Only for torsional shearing stresses, which are described by stress functions under Prandtl's soap film analog, does rod beam theory take equivalent distortions into account. In curved cross sections the relationship between the stress function and distortion leads to the determination of the curvature function across the cross section.

As long as construction used rods to provide stable and almost distortion-free structures, linear rod-beam theories were appropriate. But with the construction of aircraft and extremely light structures the situation changed. In the 1930's and 1940's, therefore, rod-beam theories were remodeled to include further effects. These particularly involved stability problems of torsion buckling and bending torsion buckling, and the tipping of beams.

The requirements of weight reduction in air- and spacecraft

construction led to lower rigidities, so that today analyses must take greater deformations into account. This has often led to taking terms of higher rank into account in cases of longitudinal strain. At the same time, however, the assumptions of classic rod-beam theory are often taken on unexamined. Now, in rod-beam theory a whole group of components of the distortion tensor are equal to zero only because terms of higher order are ignored. But if one acknowledges terms of higher order in only one component, one violates the rules of compatibility. Rod-beam theories built in this way must remain questionable, unless the validity of such an approximation can be demonstrated, at least for a certain field.

The present paper intends to derive the non-linear distortion tensor and the non-linear stress tensors in rods and beams. Subsequently, we will discuss possibile simplifications which cut off after terms of a certain higher order. It will be shown how far non-linear theories are possible in deformed rods under the hypotheses which establish a rod-beam theory as distinct from continuum theory.

In the case of the curvature of a cross section the hypothesis of flat cross sections is already modified, since one admits a deviation of parts of the cross section from the flat plane. Thus this plane establishes only a calculatory average surface of the cross section. If the deviations from this average surface remain infinitesimal, there are no extensive consequences for rod- /9 beam theory. In closed cross sections there is such a strong resistance to curving that the preconditions are established. Open cross sections are a different case. But the cross-section contours remain flat within themselves, even though the plane may no longer be perpendicular to the rod axis in the deformed state. In both cases only those expansions of the theory apply which are provided by curving force torsion theory.

Therefore we must answer the question of what expansions are

possible in the context of rod-beam theory.

The conclusion contains considerations on the determination of cutting forces, and shows that in the case of theories of higher order one must distinguish different stress definitions. Errors can especially arise when a linearized theory is applied to account for effects of the second order.

2. Basis Vectors, Distortion and Stress Tensors

The present work is founded in part on Klingbeil [8] and Fung [2]: A mass point has a pre-displacement position described by the position vector r. After displacement, which is designated by the vector v, the mass point has the position designated by the position vector R.

$$R = r + v \tag{2.1}$$

If one refers to the undistorted system as a reference state, then r is determined by the linearly independent coordinates x^i. But one can also choose the deformed state as a reference system. In this case R is described by the linearly independent coordinates y^i. As a basic system we choose a cartesian coordinate system with the unity vectors e_i.

$$v = v^i e_i \tag{2.2}$$

$$r = x^i e_i = (y^i - v^i) e_i \tag{2.3}$$

$$R = (x^i + v^i) e_i = y^i e_i \; . \tag{2.4}$$

Between the coordinates x^i and y^i there is the following /10
connection via the components of the vector of displacement:

$$y^i = x^i + v^i \tag{2.5}$$

$$x^i = y^i - v^i . \tag{2.6}$$

The differentials are:

$$dy^j = \frac{\partial y^j}{\partial x^i} dx^i = a_i^j \, dx^i = \left(\delta_i^j + \frac{\partial v^j}{\partial x^i}\right) dx^i \tag{2.7}$$

$$dx^j = \frac{\partial x^j}{\partial y^i} dy^i = b_i^j \, dy^i = \left(\delta_i^j - \frac{\partial v^j}{\partial y^i}\right) dy^i . \tag{2.8}$$

By mutual insertion of (2.7) and (2.8) into each other and with the requirement of linear independence of the coordinates, one gets the relationships which demonstrate a_i^j and b_i^j as inverse to each other.

$$\delta_k^j = a_i^j \, b_k^i = b_i^j \, a_k^i . \tag{2.9}$$

In elasticity theory two distortion tensors are known. The most used is Green's tensor, which refers distortions to the undeformed state. The counterpart is a tensor that refers distortions to the deformed state. It is also called the Almansi tensor.

First we will show the derivation by differentiation according to the coordinates of the undeformed state. The basis vectors are obtained by differentiation according to the coordinates:

$$\begin{aligned} g_i &= \frac{\partial r}{\partial x^i} = \frac{\partial x^j}{\partial x^i} e_j = \delta_i^j e_j = e_i \\ G_i &= \frac{\partial R}{\partial x^i} = \frac{\partial}{\partial x^i} (x^j + v^j) e_j = a_i^j e_j . \end{aligned} \tag{2.10}$$

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The metric coefficients result from scalar products.

$$\begin{aligned} g_{ij} &= g_i \cdot g_j = \delta_{ij} \\ G_{ij} &= G_i \cdot G_j = a_i^k a_j^k . \end{aligned} \tag{2.11}$$

From the transformations:

$$\begin{aligned} g_i &= g_{ik} g^k \\ G_i &= G_{ik} G^k \end{aligned} \tag{2.12}$$

one gets the contravariant basis vectors. Here the transformation is performed with (2.9) in the case of basis vectors of the deformed state.

$$g^i = e_i \qquad (2.13)$$
$$G^i = b_j^i e_j \, .$$

With the metric coefficients one can describe the arc elements dR and dr.

$$dR = G_i \, dx^i \qquad dR^2 = G_{ij} \, dx^i dx^j \qquad (2.14)$$
$$dr = g_i \, dx^i \qquad dr^2 = g_{ij} \, dx^i dx^j \, .$$

The components of the distortion tensor result from the differences.

$$\gamma_{ij} = \frac{1}{2} \frac{dR^2 - dr^2}{dx^i \, dx^j} = \frac{1}{2} \left(G_{ij} - g_{ij} \right) . \qquad (2.15)$$

An equivalent derivation results from differentiation according to the coordinates of the deformed state.

$$\bar{g}_i = \frac{\partial r}{\partial y^i} = \frac{\partial}{\partial y^i} \left(y^j - v^j \right) e_j = b_i{}^j e_j \qquad (2.16)$$
$$\bar{G}_i = \frac{\partial R}{\partial y^i} = \frac{\partial y^j}{\partial y^i} e_j = \delta_i{}^j e_j = e_i \; .$$

The metric coefficients: /12

$$\bar{g}_{ij} = \bar{g}_i \cdot \bar{g}_j = b_i{}^k b_j{}^k \qquad (2.17)$$
$$\bar{G}_{ij} = \bar{G}_i \cdot \bar{G}_j = \delta_{ij} \, .$$

The contravariant basis vectors:

$$\bar{g}^i = a^i_j e_j \tag{2.18}$$
$$\bar{G}^i = e_i .$$

The arc elements referring to the deformed state:

$$d\bar{R} = \bar{G}_i \, dy^i \qquad d\bar{R}^2 = \bar{G}_{ij} \, dy^i dy^j \tag{2.19}$$
$$d\bar{r} = \bar{g}_i \, dy^i \qquad d\bar{r}^2 = \bar{g}_{ij} \, dy^i dy^j .$$

The components of this tensor of distortion:

$$\bar{\gamma}_{ij} = \frac{1}{2} \; \frac{d\bar{R}^2 - d\bar{r}^2}{dy^i \, dy^j} = \frac{1}{2} (\bar{G}_{ij} - \bar{g}_{ij}) . \tag{2.20}$$

In (2.15) one finds the components of Green's distortion tensor, and in (2.20) those according to Almansi. But a tensor is fully known only when given in connection with the basis. Then it must also be irrelevant what method is used to derive the components.

Almansi's distortion tensor is referred to the deformed basis, so that the complete notation looks as follows. With equations (2.13), (2.28) and (2.9) one can show identity for both derivations.

$$\Gamma_A = \gamma_{ij} G^i G^j = \frac{1}{2} (G_{ij} - g_{ij}) G^i G^j = \frac{1}{2} (a^k_i a_j{}^k - \delta_{ij}) G^i G^j = \tag{2.21}$$
$$= \frac{1}{2} (\delta_{lm} - b^j_l b^j_m) e_l e_m =$$

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$$= \bar{\gamma}_{lm} \bar{G}^l \bar{G}^m = \frac{1}{2} (\bar{G}_{lm} - \bar{g}_{lm}) \bar{G}^l \bar{G}^m = \frac{1}{2} (\delta_{lm} - b^j_l b^j_m) e_l e_m . \tag{2.21}$$

The same applies to Green's distortion tensor, which refers to the undeformed tensor.

$$\Gamma_G = \gamma_{ij}\, g^i g^j = \frac{1}{2}(a_{ij} - g_{ij})\, g^i g^j = \frac{1}{2}(a_i{}^k a_j{}^k - \delta_{ij})\, e_i e_j =$$

$$\text{(2.22)} \qquad = \bar{\gamma}_{mn}\, \bar{g}^m \bar{g}^n = \frac{1}{2}(\bar{a}_{mn} - \bar{g}_{mn})\, \bar{g}^m \bar{g}^n = \frac{1}{2}(\delta_{mn} - b_m^k b_n^k)\, \bar{g}^m \bar{g}^n =$$

$$= \frac{1}{2}(a_i{}^k a_j{}^k - \delta_{ij})\, e_i e_j .$$

There are also equivalent definitions for the stress tensors. First there is Euler's tensor, which refers the stresses to the deformed state. Then there are Lagrange and Kirchhoff's tensors. Both refer the stresses to the undeformed state. However, they differ from the Euler tensor in their transformation laws. Lagrange's tensor does indeed refer the stresses to the undeformed state, but they act in the direction of the normal lines of the deformed system. On the other hand, the Kirchhoff tensor represents a complete transformation into the undeformed state.

(2.23) EULER:

$$S_E = \tau^{ij} G_i G_j = \bar{\tau}^{kl} \bar{G}_k \bar{G}_l$$

with

$$\bar{\tau}^{kl} = \tau^{ij} a_i{}^k a_j{}^l .$$

(2.24) LAGRANGE:

$$S_L = T^{ij} g_i g_j = \bar{T}^{kl} \bar{g}_k \bar{g}_l$$

with

$$T^{ij} = T^{kl} b_k^i b_l^j .$$

(2.25) KIRCHHOFF:

$$S_K = S^{ij} g_i g_j = S^{kl} \bar{g}_k \bar{g}_l$$

with

$$S^{ij} = \bar{S}^{kl} b_k^i b_l^j .$$

It is common practice to mention only the components. Accordingly, $\tilde{\tau}^{kl}$ are called Euler stresses, T^{ij} Lagrange stresses and S^{ij} Kirchhoff stresses. /14

The forces acting on the surface elements dF_i are as follows for the three tensors:

$$(2.26) \qquad (t^i dF_i)_E = \bar{\tau}^{ik} e_k \, \varepsilon_{ilm} \, dy^l dy^m = \sqrt{G} \, \tau^{rs} G_s \, \varepsilon_{rpq} \, dx^p dx^q$$

$$(2.27) \qquad (t^i dF_i)_L = T^{ij} e_j \, \varepsilon_{ilm} \, dx^l dx^m$$

$$(2.28) \qquad (t^i dF_i)_K = S^{ij} e_j \, \varepsilon_{ilm} \, dx^l dx^m$$

$$\varepsilon_{ilm} = \text{Permutation}.$$

The scalar triple product of the basis vectors reproduces the change in volume compared to the reference system with the unity vector. Since the mass remains unchanged, these values must be inversely proportional to the ratio of the mass density.

$$(2.29) \qquad \begin{aligned} \sqrt{G} &= (G_2 \times G_3) \cdot G_1 = \det |a_i^j| = \frac{\rho_0}{\rho} \\ \sqrt{\bar{g}} &= (\bar{g}_2 \times \bar{g}_3) \cdot \bar{g}_1 = \det |b_i^j| = \frac{\rho}{\rho_0} . \end{aligned}$$

In the Lagrange equation the forces are set equal to each other since they have the same directions.

$$(2.30) \qquad (t^i dF_i)_E = (t^i dF_i)_L .$$

From equations (2.23) to (2.30) one can derive the connections between the components by mutual substitution.

$$(2.31) \quad \text{or} \qquad \begin{aligned} T^{ik} &= \sqrt{G} \, b_j^i \, \bar{\tau}^{jk} = \sqrt{G} \, \tau^{il} a_l^k \\ \bar{\tau}^{jk} &= \frac{\rho}{\rho_0} a_i^j \, T^{ik}. \end{aligned}$$

In the Kirchhoff relationship the force referred to the

deformed system is transformed into the undeformed system. The differing directions must then be taken into account. /15

$$b_i^{*} \sqrt{G}\, \tau^{il} G_k\, \varepsilon_{lpq}\, dx^p dx^q = S^{il} e_j\, \varepsilon_{ipq}\, dx^p dx^q. \tag{2.32}$$

From this one gets the transformation for the Kirchhoff components. They must correspond to the completely transformed tensor components with the change in volume taken into account.

$$S^{ij} = \sqrt{G}\, b_k^i\, b_l^j\, \bar{\tau}^{lk} = \sqrt{G}\, \tau^{ij} \tag{2.33}$$

or

$$\bar{\tau}^{ij} = \frac{\rho}{\rho_0}\, a_l^i\, a_k^j\, S^{lk}.$$

The transformations in (2.31) to (2.33) are found thus in Fung [2]. The derivation there is based solely on considerations of differential geometry, since Fung operates only with the components of the stress tensors and thus cannot apply equations (2.23) through (2.25). As can be seen, the components of the Euler and Kirchhoff tensors are symmetrical. The components of the Lagrange tensor, on the other hand, are not symmetrical.

The Euler tensor (2.23) will be used here for the further treatment of the non-linear problem. The stress-strain relationships for the general case in any kind of curved system are derived by Green and Zerna [3].

$$\tau^{ij} = \mu\left(G^{ik} G^{jl} + G^{il} G^{jk} + \frac{2\nu}{1-2\nu}\, G^{ij} G^{kl}\right)\gamma_{kl}. \tag{2.34}$$

If one transforms the components τ^{ij} and $_{kl}$ into the components referred to the deformed system, one gets the known equations:

(2.35) and

$$\bar{\tau}^{kl} = \tau^{ij} a_i^k a_j^l$$

$$\bar{\gamma}_{kl} = \gamma_{ij} b_k^i b_l^j$$

(2.36) and

$$\bar{\tau}^{nn} = 2\mu \left(\bar{\gamma}_{nn} + \frac{\nu}{1-2\nu} \bar{\gamma}_{ij} \delta^{ij}\right)$$

$$\bar{\tau}^{ij} = 2\mu \bar{\gamma}_{ij} \quad i \neq j$$

μ = torsion modulus /16

ν = coefficient of transverse strain.

Since the deformed state of the rod is to be described only as a function of length, the position of the deformed rod axis is of fundamental importance. The position vector of a rod axis point has been described in detail by the author elsewhere [10], so that we may dispense with a detailed presentation here. The point of the rod axis in an undeformed state,

$$r_S = x e_1 , \tag{2.37}$$

converts in the deformed state into

$$R_S = (x+u)e_1 + v e_2 + w e_3 . \tag{2.38}$$

With the longitudingal element

$$ds = \sqrt{(1+u')^2 + v'^2 + w'^2}\, dx = (1+\bar{u}')\, dx \tag{2.39}$$

one gets the tangent to the spatial curve of the deformed rod axis.

$$\frac{dR_S}{ds} = \hat{e}_1 = \frac{(1+u')e_1 + v'e_2 + w'e_3}{\sqrt{(1+u')^2 + v'^2 + w'^2}} . \tag{2.40}$$

The other two unity vectors $\hat{e}_2$ and $\hat{e}_3$ are given in [10]. They are determined from the theory of curvatures of curves in space.

$$(2.41) \qquad \frac{d}{ds}\begin{Bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{Bmatrix} = \begin{pmatrix} 0 & \rho_3 & -\rho_2 \\ -\rho_3 & 0 & \rho_1 \\ \rho_2 & -\rho_1 & 0 \end{pmatrix} \begin{Bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{Bmatrix} .$$

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Here ρ_1 is the torsion corresponding to the change in angle per unit of length with reference to the rotation of the cross section around the rod axis.

$$(2.42) \qquad \rho_1 = \frac{\bar{\theta}'}{1-\bar{u}'} .$$

The curvatures ρ_2 and ρ_3 are bending curvatures determined mainly by the displacements v and w.

For the displacement of a point in the cross section, first a general statement will be made, so that the specific limitations of rod-beam theory can be made clear.

The cross-section point in the undeformed state,

$$(2.43) \qquad r = x e_1 + \eta e_2 + \xi e_3 ,$$

converts into the deformed position

$$(2.44) \qquad R = R_S + (f_1 + \bar{u}_W)\hat{e}_1 + (\eta + f_2)\hat{e}_2 + (\xi + f_3)\hat{e}_3 .$$

The f_i functions are at first still unknown. The curvature of the cross section is proportional to the torsion and a curvature function as a quality of the cross section.

$$\bar{u}_w = \varrho_1 \varphi_3 \quad . \tag{2.45}$$

The basis vectors of a deformed cross-section fiber are now determined by differentiation.

$$\begin{aligned} G_1 = \frac{\partial R}{\partial x} = (1+\bar{u}')\Big\{ & \Big[1-(\eta+f_2)\varrho_3 + (\xi+f_3)\varrho_2 + \frac{f_1'+\bar{u}_w'}{1+\bar{u}'}\Big]\hat{e}_1 + \\ & + \Big[-(\xi+f_3)\varrho_1 + (f_1+\bar{u}_w)\varrho_3 + \frac{f_2'}{1+\bar{u}'}\Big]\hat{e}_2 + \\ & + \Big[(\eta+f_2)\varrho_1 - (f_1+\bar{u}_w)\varrho_2 + \frac{f_3'}{1+\bar{u}'}\Big]\hat{e}_3\Big\} \end{aligned} \tag{2.46}$$

$$\begin{aligned} G_2 &= \frac{\partial R}{\partial \eta} = \left(\frac{\partial f_1}{\partial \eta} + \frac{\partial \bar{u}_w}{\partial \eta}\right)\hat{e}_1 + \left(1+\frac{\partial f_2}{\partial \eta}\right)\hat{e}_2 + \frac{\partial f_3}{\partial \eta}\hat{e}_3 \\ G_3 &= \frac{\partial R}{\partial \xi} = \left(\frac{\partial f_1}{\partial \xi} + \frac{\partial \bar{u}_w}{\partial \xi}\right)\hat{e}_1 + \frac{\partial f_2}{\partial \xi}\hat{e}_2 + \left(1+\frac{\partial f_3}{\partial \xi}\right)\hat{e}_3 \ . \end{aligned} \tag{2.46}$$

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These equations represent equations (2.10) for the deformed rod as a continuum. The $\hat{e}_i$ basis vectors, however, are not the basis vectors of the fundamental system e_j. But there exists only a rotation between the $\hat{e}_i$ vectors that form the accompanying trihedral for the deformed rod axis, and the e_j vectors. Thus on the basis of $\hat{e}_i$, too, the same connections as above can be derived for the stress and distortion tensors. For both stresses and distortions, the same transformations apply as between the basis systems $\hat{e}_i$ and e_j, since both bases, as orthonormalized vector systems, are identical to their contravariant bases.

This establishes the foundation for a further discussion of possible rod-beam theories. As can be seen from the basis vectors (2.40), a complete nonlinear treatment in its further expansion would become extraordinarily extensive. It would moreover be pointless, if it can be shown that only to a certain degree can non-linearities be taken into account in a rod-beam theory without contradiction.

3. Linearized Rod-Beam Theory

In a linearized theory all deformation quantities are taken into account only when they occur linearly. Since they may be assumed to be infinitesimal, elements of higher rank are negligibly small. The basis vectors in the deformed state are as follows:

$$
\begin{aligned}
G_1 &= [1+\bar{u}' - \eta\rho_3 + \xi\rho_2 + \theta''\varphi_s + f_1']\hat{e}_1 + [-\xi\theta' + f_2']\hat{e}_2 + [\eta\theta' + f_3']\hat{e}_3 \\
G_2 &= \left(\frac{\partial f_1}{\partial\eta} + \theta'\frac{\partial\varphi_s}{\partial\eta}\right)\hat{e}_1 + \left(1+\frac{\partial f_2}{\partial\eta}\right)\hat{e}_2 + \frac{\partial f_3}{\partial\eta}\hat{e}_3 \qquad (3.1)\\
G_3 &= \left(\frac{\partial f_1}{\partial\xi} + \theta'\frac{\partial\varphi_s}{\partial\xi}\right)\hat{e}_1 + \frac{\partial f_2}{\partial\xi}\hat{e}_2 + \left(1+\frac{\partial f_3}{\partial\xi}\right)\hat{e}_3 .
\end{aligned}
$$

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The linearized metric coefficients are calculated with (2.11), dropping the non-linear elements.

$$
\begin{aligned}
G_{11} &= 1+2\left(\bar{u}' - \eta\rho_3 + \xi\rho_2 + \theta''\varphi_s + f_1'\right) \\
G_{22} &= 1+2\,\frac{\partial f_2}{\partial\eta} \\
G_{33} &= 1+2\,\frac{\partial f_3}{\partial\xi} \\
G_{12} &= \theta'\left(\frac{\partial\varphi_s}{\partial\eta} - \xi\right) + \frac{\partial f_1}{\partial\eta} + f_2' \qquad (3.2)\\
G_{13} &= \theta'\left(\frac{\partial\varphi_s}{\partial\xi} + \eta\right) + \frac{\partial f_1}{\partial\xi} + f_3' \\
G_{23} &= \frac{\partial f_2}{\partial\xi} + \frac{\partial f_3}{\partial\eta} .
\end{aligned}
$$

Since the metric coefficients in the undeformed state are equal to the components of the unity vector, the distortions in (2.15) result very simply.

$$
\begin{aligned}
\gamma_{11} &= \bar{u}' - \eta\rho_3 + \xi\rho_2 + \theta''\varphi_s + f_1' \\
\gamma_{22} &= \frac{\partial f_2}{\partial\eta} \\
\gamma_{33} &= \frac{\partial f_3}{\partial\xi} \\
\gamma_{12} &= \frac{1}{2}\,\theta'\left(\frac{\partial\varphi_s}{\partial\eta} - \xi\right) + \frac{1}{2}\left(\frac{\partial f_1}{\partial\eta} + f_2'\right) \qquad (3.3)\\
\gamma_{13} &= \frac{1}{2}\,\theta'\left(\frac{\partial\varphi_s}{\partial\xi} + \eta\right) + \frac{1}{2}\left(\frac{\partial f_1}{\partial\xi} + f_3'\right) \\
\gamma_{23} &= \frac{1}{2}\left(\frac{\partial f_2}{\partial\xi} + \frac{\partial f_3}{\partial\eta}\right) .
\end{aligned}
$$

Since only linear elements can be taken into account, one can replace the contravariant metric coefficients in (2.34) with the components of the unity vector. /20

$$\tau^{11} = \frac{2\mu}{1-2\nu}\left[(1-\nu)\gamma_{11} + \nu\gamma_{22} + \nu\gamma_{33}\right]$$

$$\tau^{22} = \frac{2\mu}{1-2\nu}\left[\nu\gamma_{11} + (1-\nu)\gamma_{22} + \nu\gamma_{33}\right]$$

$$\tau^{33} = \frac{2\mu}{1-2\nu}\left[\nu\gamma_{11} + \nu\gamma_{22} + (1-\nu)\gamma_{33}\right]$$

$$\tau^{12} = 2\mu\gamma_{12} \tag{3.4}$$

$$\tau^{13} = 2\mu\gamma_{13}$$

$$\tau^{23} = 2\mu\gamma_{23} \,.$$

Linearization moreover leads in (2.23) to the fact that the stress components from (3.4) apply to both the deformed and undeformed state. In the linear case, the Euler, Kirchhoff and Lagrange torsions on the one hand, and the Green and Almansi torsions, on the other hand, are equal.

Now, with the exception of curvature, rod-beam theory requires flatness of the cross sections. But for this and all further cases of possible rod-beam theories, this means f_1 must be zero. Moreover, no normal stresses transverse to the rod axis should appear in the rods. Thus transverse contraction must be unhindered. But the cross section form must also be maintained. Thus one must require that the distortion γ_{23} become zero. Finally the shearing deformations due to bending should be negligible or zero.

$$(a) \quad f_1 = 0$$

$$(b) \quad \tau^{22} = \tau^{33} = 0$$

$$(c) \quad \gamma_{23} = \frac{1}{2}\left(\frac{\partial f_2}{\partial \xi} + \frac{\partial f_3}{\partial \eta}\right) = 0 \tag{3.5}$$

$$(d) \quad \begin{aligned} &\frac{\partial f_1}{\partial \eta} + f_2' \approx 0 \\ &\frac{\partial f_1}{\partial \xi} + f_3' \approx 0 \,. \end{aligned}$$

Condition (b) in (3.5) leads to the following result:

$$\tau'' = \frac{2\mu}{1-2\nu}(1-\nu-2\nu^2)\gamma_{11} = 2\mu(1+\nu)\gamma_{11} = E\gamma_{11} \tag{3.6}$$

$$\gamma_{22} = \gamma_{33} = -\nu\gamma_{11} = -\nu\left[\bar{u}' - \eta\varrho_3 + \xi\varrho_2 + \theta''\varphi_s\right] = \frac{\partial f_2}{\partial \eta} = \frac{\partial f_3}{\partial \xi} \tag{3.7}$$

The hitherto unknown functions f_2 and f_3 can be determined by integration from (3.7):

$$\begin{aligned} f_2 &= -\nu\left[\bar{u}'\eta - \tfrac{1}{2}\eta^2\varrho_3 + \eta\xi\varrho_2 + \theta''\int\varphi_s\,d\eta\right] + c_2(\xi) \\ f_3 &= -\nu\left[\bar{u}'\xi - \eta\xi\varrho_3 + \tfrac{1}{2}\xi^2\varrho_2 + \theta''\int\varphi_s\,d\xi\right] + c_3(\eta) \,. \end{aligned} \tag{3.8}$$

The solutions to (3.8) must satisfy condition (c) in (3.5).

$$\begin{aligned} \frac{\partial f_2}{\partial \xi} &= -\nu\left[\eta\varrho_2 + \theta''\int\frac{\partial\varphi_s}{\partial\xi}\,d\eta\right] + \frac{\partial c_2}{\partial \xi} \\ \frac{\partial f_3}{\partial \eta} &= -\nu\left[-\xi\varrho_3 + \theta''\int\frac{\partial\varphi_s}{\partial\eta}\,d\xi\right] + \frac{\partial c_3}{\partial \eta} \,. \end{aligned} \tag{3.9}$$

The following connection exists between the stress function for the torsion stresses and the curvature function:

$$\frac{\partial\varphi_s}{\partial\eta} = \frac{\partial\Phi}{\partial\xi} + \xi \,, \quad \frac{\partial\varphi_s}{\partial\xi} = -\frac{\partial\Phi}{\partial\eta} - \eta \,. \tag{3.10}$$

With the means of partial integration one then gets from (3.9) with (3.10):

$$\frac{\partial f_2}{\partial \xi} + \frac{\partial f_3}{\partial \eta} = -\nu\left[\eta\varrho_2 - \xi\varrho_3 + \tfrac{1}{2}\theta''(\xi^2 - \eta^2)\right] + \frac{\partial c_2(\xi)}{\partial \xi} + \frac{\partial c_3(\eta)}{\partial \eta} \,. \tag{3.11}$$

To satisfy condition (c) in (3.5) we must have

$$\begin{aligned} c_2(\xi) &= \nu\left[-\tfrac{1}{2}\xi^2\varrho_3 + \tfrac{1}{6}\theta''\xi^3\right] + c_2^{\circ} \\ c_3(\eta) &= \nu\left[\tfrac{1}{2}\eta^2\varrho_2 - \tfrac{1}{6}\theta''\eta^3\right] + c_3^{\circ} \,. \end{aligned} \tag{3.12}$$

If it is predicted that the cross-sections will remain /22

flat, then $f_1 = 0$. If at the same time only torsional shearing deformations are to be taken into account, then f_2' and f_3' must be of a negligible order of magnitude. Thus one must set down a few considerations on the order of magnitude of the deformations and cross-section dimensions.

In theories of higher rank one assumes that the deformation quantities are finite but small in comparison to the length of the rod. They are viewed as small in rank one. The longitudinal strain in the direction of the deformed rod can result either when a longitudinal force acts directly on the rod, or when such a force arises because of bending when the rod ends cannot be displaced. In both cases one must consider it small in rank two, as can be seen in the description of the longitudinal element (2.39). The cross-section dimensions in a rod must likewise be small compared to the length. The maximum values of the coordinates η and ξ are thus small in rank one. The curvature function ϕ_s, because of a quantity equation in (3.10), must thus be small in rank two. Moreover, one can view the coefficient of transverse strain ν as still being small in rank one. Its maximum value of 0.5 for incompressible materials cannot be expected in rods, since such materials are rubbery and have inadequate resistance to bending. In general one must anticipate $\nu = 0.3$.

One thus gets the following hierarchy of orders of magnitude.

$$(3.13) \qquad \begin{aligned} &o(\varepsilon): \ \eta, \xi, \frac{\partial \varphi_s}{\partial \eta}, \frac{\partial \varphi_s}{\partial \xi}, \rho_2, \rho_3, \theta', \theta'', \nu \\ &o(\varepsilon^2): \ \varphi_s, \bar{u}' \ . \end{aligned}$$

Thus the largest terms in f_2 and f_3 are small in rank four.

$$(3.14) \qquad o(\varepsilon^4): \ f_2, f_3 \ .$$

Therefore a rod-beam theory with requirements (3.5) can /23
be considered free of contradictions only to the third order under (3.13). If terms of higher rank are taken into account, the normal

preconditions in the context of rod-beam theory are violated. An improved solution is thus possible only by way of disc, plate or shell theories, or in extreme cases, continuum theory.

We will include a short discussion for very thin rods in which the cross-section dimensions can be designated as small in rank two. The hierarchy of orders of magnitude then appears as follows:

$$
\begin{aligned}
&o(\varepsilon) : \varrho_2 , \varrho_3 , \theta', \theta'', \nu \\
(3.15) \qquad &o(\varepsilon^2) : \eta, \xi, \frac{\partial \varphi_s}{\partial \eta}, \frac{\partial \varphi_s}{\partial \xi}, \bar{u}' \\
&o(\varepsilon^4) : \varphi_s \; .
\end{aligned}
$$

If one wants to include terms up to the fifth order in such a theory, in order to include the curvature force effects from $\theta''\phi_s$, there is still one term each left for the functions f_2 and f_3.

$$
\begin{aligned}
(3.16) \qquad &f_2 = -\nu \bar{u}' \eta + c_2^o \mathrel{\hat{=}} o(\varepsilon^5) \\
&f_3 = -\nu \bar{u}' \xi + c_3^o \mathrel{\hat{=}} o(\varepsilon^5) \quad .
\end{aligned}
$$

Shear stresses result from this when $\bar{u}'$ is not constant throughout the length. The variable transverse contraction causes torsional distortions if the cross sections are to remain flat at the same time. But if one allows the cross section points to deviate from this hypothesis, it would be small in rank seven for very thin rods.

$$
(3.17) \qquad f_1 = \frac{1}{2} \nu \bar{u}'' (\eta^2 + \xi^2) + c_2^o \eta + c_3^o \xi + c_1^o \mathrel{\hat{=}} o(\varepsilon^7) \; .
$$

Since only terms up to the fifth order are taken into account, the cross section can be called flat with adequate precision in the context of these considerations. At the same time the shear

distortions can be further neglected, and the characteristic stress state for rods is maintained.

We have thus established the outer limits within which rod-beam theories of higher order must be realized.

4. Rod-Beam Theories of Higher Order

In this section we will draw conclusions for rods in which a finite, known torsion can appear. Such rods appear as rotor blades in helicopters or extending/retracting rod antennae with open sections in satellites. The rod theories used in these cases [1,6,7,9] are either linearized theories with equilibrium formulations in the deformed state, or non-linear theories that lack an unobjectionable distinction between stress and strain tensors. The concluding section will discuss the possible consequences of confusing Euler and Kirchhoff stresses.

4.1. Simple Rod Theory

For a rod with a finite pre-twist, torsion in the deformed state is

$$\varrho_1 = \frac{\theta_0' + \theta'}{1 + \bar{u}'} , \qquad (4.1)$$

so that for the curvature of the cross section one gets:

$$\bar{u}_w = \varrho_1 \varphi_s = \frac{\theta_0' + \theta'}{1 + \bar{u}'} \varphi_s \qquad (4.2)$$

The orders of magnitude for this case are to be assumed as follows:

$$o(\varepsilon) : \eta, \xi, \frac{\partial \varphi_s}{\partial \eta}, \frac{\partial \varphi_s}{\partial \xi}, \varrho_2, \varrho_3, \theta', \theta'', \nu \qquad (4.3)$$

$$o(\varepsilon^2) : \varphi_s, \bar{u}' \qquad (4.3)$$

$$o(\varepsilon^3) : \frac{\partial f_2}{\partial \eta}, \frac{\partial f_2}{\partial \xi}, \frac{\partial f_3}{\partial \eta}, \frac{\partial f_3}{\partial \xi} .$$

/25

Terms of higher orders than ε^3 will be neglected. From equations (2.46) one can form the basis vectors G_i up to order ε^3. The stress-free original state is merely the pre-twisted rod. The components of the distortion tensor in this case are:

$$
(4.4)\qquad
\begin{aligned}
\gamma_{11} &= \bar{u}' - \eta\rho_3 + \xi\rho_2 + \theta''\varphi_s + (\eta^2+\xi^2)\theta_0'\theta' \\
\gamma_{22} &= \left(\frac{\partial\varphi_s}{\partial\eta}\right)^2 \theta_0'\theta' + \frac{\partial f_2}{\partial\eta} \\
\gamma_{33} &= \left(\frac{\partial\varphi_s}{\partial\xi}\right)^2 \theta_0'\theta' + \frac{\partial f_3}{\partial\xi} \\
\gamma_{12} &= \frac{1}{2}\left[\theta'\left(\frac{\partial\varphi_s}{\partial\eta} - \xi\right) + \theta_0'\frac{\partial\varphi_s}{\partial\eta}\left(-\eta\rho_3 + \xi\rho_2\right) + \theta_0'\rho_3\varphi_s\right] \\
\gamma_{13} &= \frac{1}{2}\left[\theta'\left(\frac{\partial\varphi_s}{\partial\xi} + \eta\right) + \theta_0'\frac{\partial\varphi_s}{\partial\xi}\left(-\eta\rho_3 + \xi\rho_2\right) - \theta_0'\rho_2\varphi_s\right] \\
\gamma_{23} &= \frac{\partial\varphi_s}{\partial\eta}\frac{\partial\varphi_s}{\partial\xi}\theta_0'\theta' + \frac{1}{2}\left(\frac{\partial f_2}{\partial\xi} + \frac{\partial f_3}{\partial\eta}\right) .
\end{aligned}
$$

To transform the γ_{ij} components into the $\bar{\gamma}_{kl}$ components under (2.35), one needs transformation coefficients only up to rank ε. Since the lowest order of distortion is ε^2, no terms of higher order are needed for the transformation in the context of this approximation.

$$(4.5)\qquad (G_i)_N = a_i^{\,j}\hat{e}_j$$

with

$$(4.6)\qquad a_i^{\,j} = \begin{pmatrix} 1 & -\theta_0'\xi & \theta_0'\eta \\ \theta_0'\dfrac{\partial\varphi_s}{\partial\eta} & 1 & 0 \\ \theta_0'\dfrac{\partial\varphi_s}{\partial\xi} & 0 & 1 \end{pmatrix}$$ /26

and

$$(4.7)\qquad (G^i)_N = b_j^{\,i}\hat{e}^j$$

with

$$(4.8)\qquad b_j^{\,i} = \begin{pmatrix} 1 & -\theta_0'\dfrac{\partial\varphi_s}{\partial\eta} & -\theta_0'\dfrac{\partial\varphi_s}{\partial\xi} \\ \theta_0'\xi & 1 & 0 \\ -\theta_0'\eta & 0 & 1 \end{pmatrix}$$

and

$$(4.9)\qquad \left(\sqrt{G}\right)_N = 1 .$$

The $\hat{\gamma}_{kl}$ components of the Almansi tensor in the accompanying $\hat{e}_i$ rod system are:

$$
(4.10)\qquad
\begin{aligned}
\hat{\gamma}_{11} &= \bar{u}' - \eta\rho_3 + \xi\rho_2 + \theta''\varphi_s + \theta_0'\theta'\left(\xi\frac{\partial\varphi_s}{\partial\eta} - \eta\frac{\partial\varphi_s}{\partial\xi}\right)\\
\hat{\gamma}_{22} &= \frac{\partial f_2}{\partial\eta} + \theta_0'\theta'\xi\frac{\partial\varphi_s}{\partial\eta}\\
\hat{\gamma}_{33} &= \frac{\partial f_3}{\partial\xi} - \theta_0'\theta'\eta\frac{\partial\varphi_s}{\partial\xi}\\
\hat{\gamma}_{12} &= \frac{1}{2}\left[\theta'\left(\frac{\partial\varphi_s}{\partial\eta} - \xi\right) - \theta_0'\frac{\partial\varphi_s}{\partial\eta}\left(2\bar{u}' - \eta\rho_3 + \xi\rho_2\right) + \theta_0'\rho_3\varphi_s\right]\\
\hat{\gamma}_{13} &= \frac{1}{2}\left[\theta'\left(\frac{\partial\varphi_s}{\partial\xi} + \eta\right) - \theta_0'\frac{\partial\varphi_s}{\partial\xi}\left(2\bar{u}' - \eta\rho_3 + \xi\rho_2\right) - \theta_0'\rho_2\varphi_s\right]\\
\hat{\gamma}_{23} &= \frac{1}{2}\left(\frac{\partial f_2}{\partial\xi} + \frac{\partial f_3}{\partial\eta}\right) + \frac{1}{2}\theta_0'\theta'\left(\xi\frac{\partial\varphi_s}{\partial\xi} - \eta\frac{\partial\varphi_s}{\partial\eta}\right).
\end{aligned}
$$

The $\hat{\gamma}_{kl}$ and $\hat{\tau}^{kl}$ components involve distortions and stresses referred to the deformed state. The pertinent orthonormalized basis is the rod's own system, the accompanying trihedral of the rod axis $\hat{e}_i$. The assumptions of rod-beam theory must thus apply to these components. These requirements, as raised in (3.5), are satisfied by the following solution

/27

$$
(4.11)\qquad
\begin{aligned}
f_2 = -\nu\left[\bar{u}'\eta - \frac{1}{2}\eta^2\rho_3 + \eta\xi\rho_2 + \theta''\int\varphi_s\,d\eta\right] -\\
-\theta_0'\theta'\left[(1+\nu)\xi\varphi_s - \nu\int\eta\frac{\partial\varphi_s}{\partial\xi}\,d\eta\right] + c_2(\xi)\\
f_3 = -\nu\left[\bar{u}'\xi - \eta\xi\rho_3 + \frac{1}{2}\xi^2\rho_2 + \theta''\int\varphi_s\,d\xi\right] -\\
-\theta_0'\theta'\left[\nu\int\xi\frac{\partial\varphi_s}{\partial\eta}\,d\xi - (1+\nu)\eta\varphi_s\right] + c_3(\eta).
\end{aligned}
$$

This yields the $\hat{\tau}^{kl}$ components of the Euler tensor in the accompanying system $\hat{e}_i$.

$$
\begin{aligned}
&\hat{\tau}^{11} = E\left[\bar{u}' - \eta\varrho_3 + \xi\varrho_2 + \Theta''\varphi_s + \Theta_0'\Theta'\left(\xi\frac{\partial\varphi_s}{\partial\eta} - \eta\frac{\partial\varphi_s}{\partial\xi}\right)\right] \\
&\hat{\tau}^{22} = \hat{\tau}^{33} = 0 \\
(4.12)\quad &\hat{\tau}^{12} = \mu\left[\Theta'\left(\frac{\partial\varphi_s}{\partial\eta} - \xi\right) - \Theta_0'\frac{\partial\varphi_s}{\partial\eta}\left(2\bar{u}' - \eta\varrho_3 + \xi\varrho_2\right) + \Theta_0'\varrho_3\varphi_s\right] + \hat{\tau}^{12}_{(B+W)} \\
&\hat{\tau}^{13} = \mu\left[\Theta'\left(\frac{\partial\varphi_s}{\partial\xi} + \eta\right) - \Theta_0'\frac{\partial\varphi_s}{\partial\xi}\left(2\bar{u}' - \eta\varrho_3 + \xi\varrho_2\right) - \Theta_0'\varrho_2\varphi_s\right] + \hat{\tau}^{13}_{(B+W)} \\
&\hat{\tau}^{23} = 0
\end{aligned}
$$

$\hat{\tau}^{12}_{(B+W)}, \hat{\tau}^{13}_{(B+W)}$ = bending and curving shear stresses with negligible distortions.

As can clearly be seen from equations (4.12), this theory yields only stresses and distortions which are linearly dependent on displacements and twisting. Thus only a linearized theory can be carried out; but it makes it possible to set up the equilibrium in the deformed state.

4.2. Non-Linear Rod-Beam Theory

The orders of magnitude for very thin rods must be applied in this case.

/28

$$
\begin{aligned}
&o(\varepsilon):\ \varrho_2,\ \varrho_3,\ \Theta',\ \Theta'',\ \nu \\
&o(\varepsilon^2):\ \eta,\ \xi,\ \frac{\partial\varphi_s}{\partial\eta},\ \frac{\partial\varphi_s}{\partial\xi},\ \bar{u}' \\
(4.13)\quad &o(\varepsilon^3):\ \frac{\partial f_2}{\partial\eta},\ \frac{\partial f_2}{\partial\xi},\ \frac{\partial f_3}{\partial\eta},\ \frac{\partial f_3}{\partial\xi} \\
&o(\varepsilon^4):\ \varphi_s \\
&o(\varepsilon^5):\ f_2,\ f_2',\ f_3,\ f_3',\ \frac{\partial f_1}{\partial\eta},\ \frac{\partial f_1}{\partial\xi}.
\end{aligned}
$$

Terms of orders higher than ε^5 will be neglected in this theory. From equations (2.46), the basis vectors of G_i are formed up to rank ε^5. The stressless state is again the pre-twisted rod. The components of the distortion tensors are now:

$$
\begin{aligned}
\gamma_{11} &= \bar{u}' + \frac{1}{2}\bar{u}'^2 + (1+2\bar{u}')(-\eta\rho_3 + \xi\rho_2) + \theta''\varphi_3 + (\eta^2+\xi^2)\theta_0'\theta' \\
\gamma_{22} &= \left(\frac{\partial\varphi_3}{\partial\eta}\right)^2 \theta_0'\theta' + \frac{\partial f_2}{\partial\eta} \\
\gamma_{33} &= \left(\frac{\partial\varphi_3}{\partial\xi}\right)^2 \theta_0'\theta' + \frac{\partial f_3}{\partial\xi} \\
\gamma_{12} &= \frac{1}{2}\left[\theta'\left(\frac{\partial\varphi_3}{\partial\eta} - \xi\right) + \theta_0'\frac{\partial\varphi_3}{\partial\eta}(-\eta\rho_3 + \xi\rho_2) + \frac{\partial f_1}{\partial\eta} + f_2' + \right. \\
&\qquad \left. + \theta_0'\left(\varphi_3\rho_3 - f_3 - \xi\frac{\partial f_2}{\partial\eta} + \eta\frac{\partial f_3}{\partial\eta}\right)\right] \\
\gamma_{13} &= \frac{1}{2}\left[\theta'\left(\frac{\partial\varphi_3}{\partial\xi} + \eta\right) + \theta_0'\frac{\partial\varphi_3}{\partial\xi}(-\eta\rho_3 + \xi\rho_2) + \frac{\partial f_1}{\partial\xi} + f_3' + \right. \\
&\qquad \left. + \theta_0'\left(-\varphi_3\rho_2 + f_2 - \xi\frac{\partial f_2}{\partial\xi} + \eta\frac{\partial f_3}{\partial\xi}\right)\right] \\
\gamma_{23} &= \frac{\partial\varphi_3}{\partial\eta}\frac{\partial\varphi_3}{\partial\xi}\theta_0'\theta' + \frac{1}{2}\left(\frac{\partial f_2}{\partial\xi} + \frac{\partial f_3}{\partial\eta}\right).
\end{aligned}
\tag{4.14}
$$

Since the distortions present rank ε^2 as the lowest one, the transformation coefficients for the transformation into Euler and Almansi components must be developed to rank ε^3.

/29

$$(G_i)_N = a_i^j \hat{e}_j \tag{4.15}$$

with

$$
a_i^j = \begin{pmatrix}
(1+\bar{u}' - \eta\rho_3 + \xi\rho_2) & -(\theta_0'+\theta')\xi & (\theta_0'+\theta')\eta \\
(\theta_0'+\theta')\frac{\partial\varphi_3}{\partial\eta} & 1+\frac{\partial f_2}{\partial\eta} & \frac{\partial f_3}{\partial\eta} \\
(\theta_0'+\theta')\frac{\partial\varphi_3}{\partial\xi} & \frac{\partial f_2}{\partial\xi} & 1+\frac{\partial f_3}{\partial\xi}
\end{pmatrix}
\tag{4.16}
$$

and

$$(G^i)_N = b_j^i \hat{e}^j \tag{4.17}$$

with

$$(4.18)\qquad b^i_j = \begin{pmatrix} (1-\bar{u}'+\eta\rho_3-\xi\rho_2) & -(\theta_0'+\theta')\dfrac{\partial\varphi_s}{\partial\eta} & -(\theta_0'+\theta')\dfrac{\partial\varphi_s}{\partial\xi} \\ (\theta_0'+\theta')\xi & 1-\dfrac{\partial f_2}{\partial\eta} & -\dfrac{\partial f_2}{\partial\xi} \\ -(\theta_0'+\theta')\eta & -\dfrac{\partial f_3}{\partial\eta} & 1-\dfrac{\partial f_3}{\partial\xi} \end{pmatrix}$$

and

$$(4.19)\qquad (\sqrt{G})_N = 1+\bar{u}'-\eta\rho_3+\xi\rho_2+\frac{\partial f_2}{\partial\eta}+\frac{\partial f_3}{\partial\xi}\,.$$

The $\hat{\gamma}_{kl}$ coefficients become

$$\hat{\gamma}_{11} = \bar{u}'-\frac{3}{2}\bar{u}'^2+(1-2\bar{u}')(-\eta\rho_3+\xi\rho_2)+\theta''\varphi_s+\theta_0'\theta'\left(\xi\frac{\partial\varphi_s}{\partial\eta}-\eta\frac{\partial\varphi_s}{\partial\xi}\right)$$

$$\hat{\gamma}_{22} = \frac{\partial f_2}{\partial\eta}+\theta_0'\theta'\xi\frac{\partial\varphi_s}{\partial\eta}$$

$$\hat{\gamma}_{33} = \frac{\partial f_3}{\partial\xi}-\theta_0'\theta'\eta\frac{\partial\varphi_s}{\partial\xi}$$

$$(4.20)\qquad \hat{\gamma}_{12} = \frac{1}{2}\left[\theta'\left(\frac{\partial\varphi_s}{\partial\eta}-\xi\right)(1-\bar{u}')-2\theta'\bar{u}'\frac{\partial\varphi_s}{\partial\eta}-\theta_0'\frac{\partial\varphi_s}{\partial\eta}(2\bar{u}'-\eta\rho_3+\xi\rho_2)+\right.$$
$$\left.+\frac{\partial f_1}{\partial\eta}+f_2'+\theta_0'\left(\varphi_s\rho_3-f_3+\xi\frac{\partial f_2}{\partial\eta}-\eta\frac{\partial f_2}{\partial\xi}\right)\right]$$

$$\hat{\gamma}_{13} = \frac{1}{2}\left[\theta'\left(\frac{\partial\varphi_s}{\partial\xi}+\eta\right)(1-\bar{u}')-2\theta'\bar{u}'\frac{\partial\varphi_s}{\partial\xi}-\theta_0'\frac{\partial\varphi_s}{\partial\xi}(2\bar{u}'-\eta\rho_3+\xi\rho_2)+\right.$$ /30
$$(4.20)\qquad \left.+\frac{\partial f_1}{\partial\xi}+f_3'+\theta_0'\left(-\varphi_s\rho_2+f_2+\xi\frac{\partial f_3}{\partial\eta}-\eta\frac{\partial f_3}{\partial\xi}\right)\right]$$

$$\hat{\gamma}_{23} = \frac{1}{2}\left(\frac{\partial f_2}{\partial\xi}+\frac{\partial f_3}{\partial\eta}\right)+\frac{1}{2}\theta_0'\theta'\left(\xi\frac{\partial\varphi_s}{\partial\xi}-\eta\frac{\partial\varphi_s}{\partial\eta}\right).$$

The requirements of (3.5) are satisfied by the following solution:

$$f_2 = -\nu\left[\left(\bar{u}'-\frac{3}{2}\bar{u}'^2\right)\eta+(1-2\bar{u}')\left(-\frac{1}{2}\eta^2\rho_3+\eta\xi\rho_2\right)+\theta''\int\varphi_s\,d\eta\right]-$$
$$-\theta_0'\theta\left[(1+\nu)\xi\varphi_s-\nu\int\eta\frac{\partial\varphi_s}{\partial\xi}d\eta\right]+c_2(\xi)$$

$$(4.21)$$

$$f_3 = -\nu\left[\left(\bar{u}'-\frac{3}{2}\bar{u}'^2\right)\xi+(1-2\bar{u}')\left(-\eta\xi\rho_3+\frac{1}{2}\xi^2\rho_2\right)+\theta''\int\varphi_s\,d\xi\right]-$$
$$-\theta_0'\theta'\left[\nu\int\xi\frac{\partial\varphi_s}{\partial\eta}d\xi-(1+\nu)\eta\varphi_s\right]+c_3(\eta).$$

In order to be able to ignore the shearing deformations further, a negligible function of rank ε^γ is assumed for f_1 as in (3.17). The $\hat{\tau}^{kl}$ components of the Euler tensor result as follows:

$$
\begin{aligned}
\hat{\tau}^{11} &= E\left[\bar{u}' - \frac{3}{2}\bar{u}'^2 + (1-2\bar{u}')(-\eta\rho_3 + \xi\rho_2) + \theta''\varphi_s + \theta_0'\theta'\left(\xi\frac{\partial\varphi_s}{\partial\eta} - \eta\frac{\partial\varphi_s}{\partial\xi}\right)\right] \\
\hat{\tau}^{12} &= \mu\left[\theta'\left(\frac{\partial\varphi_s}{\partial\eta} - \xi\right)(1-\bar{u}') - 2\bar{u}'\theta'\frac{\partial\varphi_s}{\partial\eta} - \theta_0'\frac{\partial\varphi_s}{\partial\eta}(2\bar{u}' - \eta\rho_3 + \xi\rho_2) + \theta_0'\varphi_s\rho_3\right] + \\
&\qquad + \hat{\tau}^{12}_{(B+W)} \\
\hat{\tau}^{13} &= \mu\left[\theta'\left(\frac{\partial\varphi_s}{\partial\xi} + \eta\right)(1-\bar{u}') - 2\bar{u}'\theta'\frac{\partial\varphi_s}{\partial\xi} - \theta_0'\frac{\partial\varphi_s}{\partial\xi}(2\bar{u}' - \eta\rho_3 + \xi\rho_2) - \theta_0'\varphi_s\rho_2\right] + \\
&\qquad + \hat{\tau}^{13}_{(B+W)} \\
\hat{\tau}^{22} &= \hat{\tau}^{33} = \hat{\tau}^{23} = 0.
\end{aligned}
\tag{4.22}
$$

The stresses in (4.22) make possible a rod-beam theory that allows nonlinear terms in the curvatures ρ_2 and ρ_3. A theory to rank ε^5 as in (4.13) represents the most extreme degree of non-linearity in a general theory for very thin rods. For special, less general stresses, however, the possibilities for individual cases are not so limited. But in most cases a general stress with longitudinal force, bending and torsion cannot be /31
ruled out.

5. Negligibility of Secondary Shear Deformations

In the previous sections it was shown how a compatible state of deformation can be brought into accord with rod hypotheses by the defined neglect of terms of higher rank. But secondary stresses also occur in rods for reasons of equilibrium, when the longitudinal stresses in the rod are not constant across its length due to variable bending or curvature.

The resulting shear distortions are not taken into account in the deformation expression for the rod, as a result of the simplifying hypotheses. These shear deformations thus violate the compatibility conditions. A significant problem of all rod theories is therefore to make these additional deformations "tolerable". For the case of thick rods, approximations have been developed and show the remaining error to be negligible in most cases [4,5,11].

Here we will consider only thin rods. Here again we find the question of what degree of precision still makes sense in a non-linear theory, when the secondary shear deformations are to be further neglected, as is generally the case. If a rod theory takes account of terms of higher order, of the same order of magnitude as the neglected shear deformations, the theory becomes dubious.

If one turns to linear theories, initial equations of order of magnitude can be given for the secondary shear deformations. Since the linear theory does not have to distinguish between the different definitions of stresses and distortions, the equilibrium in the longitudinal direction can be written directly with the components from (3.4). Here additional expressions must be taken into account for shear stresses due to variable longitudinal stresses.

$$\text{(5.1)} \qquad \frac{\partial \tau^{11}}{\partial x} + \frac{\partial(\tau^{21} + \tau^{21}_{B+W})}{\partial \eta} + \frac{\partial(\tau^{31} + \tau^{31}_{B+W})}{\partial \xi} + q_x = 0 .$$

Equation (5.1) can be transformed with (3.3), (3.4), (3.8) /32
and (3.12).

$$\text{(5.2)} \qquad (E - 2\mu\nu)\left[\bar{u}'' - \eta \varrho_3' + \xi \varrho_2' + \theta''' \varphi_S\right] + \frac{\partial \tau^{21}_{B+W}}{\partial \eta} + \frac{\partial \tau^{31}_{B+W}}{\partial \xi} + q_x = 0 .$$

In equation (5.2) the f_2 and f_3 solutions from Section 3 are still taken into account, since their derivations from the cross-section coordinates are used. The resulting stresses and distortions remain of the same order of magnitude as f_2 and f_3, and are thus negligible in the context described above.

From equation (5.2) one can split off an equation for variable longitudinal force.

$$\text{(5.3)} \qquad E\bar{u}'' + q_x = 0 .$$

Here the alteration of the longitudinal force, for instance, is proportional to the inertial forces, due to the rod's weight, which are constant across the cross section. Any alterations of q_x across the cross section, whether linear or proportional to curvature, would contain bending and curvature inertia terms. They are of the same order of magnitude as the corresponding terms from the longitudinal stress, so that one can dispense with an explicit acknowledgement of them.

Also conceivable is an alteration in longitudinal force due to shear stress on the rod surface in the longitudinal direction. The resulting force introduction problem, however, cannot be solved with the tools of rod-beam theory. Strictly speaking, external transverse loads are also not possible in rods, since they would entail stresses normal to the rod axis.

If, however, it is possible to represent secondary shear deformations as negligible in rod-beam theory, then the rod-beam theory in itself is justified. The equilibrium in a single volume element of the rod then becomes uninteresting; only the equilibrium in a rod segment is set up. Here the cutting forces integrated across the cross section are used.

Taking (5.3) and the negligibility of the transverse deformations into account, one gets the secondary shear stresses as follows: /33

$$\tau^{21}_{B+W} = E\varrho_3' \int_{\eta_R}^{\eta} \eta \, d\eta + E\Theta''' \frac{\partial \psi}{\partial \eta},$$

$$\tau^{31}_{B+W} = - E\varrho_2' \int_{\xi_R}^{\xi} \xi \, d\xi + E\Theta''' \frac{\partial \psi}{\partial \xi} \tag{5.4}$$

with

$$\frac{\partial^2 \psi}{\partial \eta^2} + \frac{\partial^2 \psi}{\partial \xi^2} = - \varphi_S .$$

With the order of magnitude in (3.13) one gets

$$(5.5) \qquad \theta''' \frac{\partial \psi}{\partial \eta} = o(\varepsilon^4) \text{ and } \theta''' \frac{\partial \psi}{\partial \xi} = o(\varepsilon^4).$$

The other terms in (5.4) seem to be of the third order. But one must note that the shear stresses can only run parallel to the outer edges of a cross section. Perpendicular to the edges they must be zero. The parabola-shaped curve thus caused further limits the size of the shear deformation.

For a rectangle of height h with $-h/2 < \eta \leq +h/2$ and $h = o(\varepsilon)$ one gets

$$(5.6) \qquad \gamma_{12} = \frac{E}{2\mu} \rho_3' \left(\frac{h^2}{8} - \frac{\eta^2}{2}\right).$$

The maximum for γ_{12} at $2\mu \approx E$ is then

$$(5.7) \qquad \max \gamma_{12} = \rho_3' \frac{h^2}{8} = o(\varepsilon^4).$$

Thus in thin rods one can assume that the secondary shear distortions can be considered to be of the fourth order. The conclusions drawn at the end of Section 3 remain the same, even when one is considering very thin rods.

In the case of theories of higher order one can provide the same demonstration for each individual case. For the statement /34 of equilibrium (5.1) the Lagrange components referred to the basis system e_i are are most suitable [2]. The equation then takes the same form.

$$(5.8) \qquad \frac{\partial T^{11}}{\partial x} + \frac{\partial(T^{21} + T^{21}_{B \cdot w})}{\partial \eta} + \frac{\partial(T^{31} + T^{31}_{B \cdot w})}{\partial \xi} + q_x = 0$$

The only tiresome part here is the many transformations, where one must also take into account the fact that the Lagrange tensor is not symmetrical.

If one considers only the secondary shear formations, the degree of precision can be further increased by suitable approximations. But, as shown in the last Section, one would thus get into orders of magnitude in which the neglected transverse strains would again play a role. Thus only special applications of theories of higher orders are conceivable. A general, non-linear rod-beam theory, on the other hand, remains valid only to a limited extent.

6. Cutting Forces

Within theories of higher order one must set up the equilibrium in the deformed body. For rods this means that the cutting forces and moments must be determined for the deformed rod. Since the surface differentials which are necessary in connection with the Euler tensor $\bar{\tau}^{kl}$ under (2.26) are either unknown or hard to come by, one must detour to the Lagrange components under (2.31). Under definition (2.30) this does not change the size and direction of the cutting forces.

The stress vector in the rod cross section, referred to the undeformed cross section in the direction of the normal line of a deformed surface differential, is thus formed according to (2.31).

$$(6.1) \qquad \hat{T}^{1k}\hat{e}_k = \sqrt{G}\, b_j^1\, \hat{\tau}^{jk}\hat{e}_k = \sqrt{G}\, \tau^{1l} a_l^k \hat{e}_k \ .$$

In linear theory both the volume changes and the transformation a_l^k and b_i^j are neglected. Thus no distinction is made among Euler, Lagrange and Kirchhoff stresses. However, it is frequently the case that one must consider known or partially known stresses on the rod. Since in this case the deformations by way of the stress-strain relationship can be replaced by the corresponding /35
forces, it is possible within a linearized theory to take transformations into account in (6.1) to a certain extent. In the case of such a "postponed" expansion of the theory to a higher order

-- equilibrium in the defined rod -- the stresses are often confused. As soon as the classic linear theory is left behind, one must distinguish Euler, Lagrange and Kirchhoff stresses.

With a simple example we will show shat errors can be produced by a non-observance of the different stress definitions. We choose the example of a curvature-free cross section ($\psi = 0$) under torsion and longitudinal stress. A completely curvature-free cross section is for instance a closed circular cross section in which the shearing axis center and the centroidal axis are the same. A precision of ε^2 is to be reached under the simple rod-beam theory (Sec. 4.1). There will be no pre-twisting of the cross section.

Under (4.12) the Euler components as a function of the deformations $\bar{u}'$ and $\bar{\theta}'$ are as follows:

$$
\begin{aligned}
&\hat{\tau}^{11} = E\bar{u}' \\
&\hat{\tau}^{22} = \hat{\tau}^{33} = 0 \\
(6.2)\qquad &\hat{\tau}^{12} = -\mu\xi\theta' \\
&\hat{\tau}^{13} = \mu\eta\theta' \\
&\hat{\tau}^{23} = 0
\end{aligned}
$$

If one stays with strict linear theory, then $\hat{\tau}^{ij} = \hat{S}^{ij} = \hat{T}^{ij}$, since the coefficients of transformation, insofar as they are a function of the deformations, are still negligible. But if one assumes that the stresses can be expressed by known forces, then despite linear theory, the transformations can take into account those terms that contain the distortions linearly. One develops a linearized theory with the establishment of equilibrium in the deformed rod.

From (2.46) the basis vectors to rank ε^2 are obtained:

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(6.3) $$G_i = a_i^j \hat{e}_j$$

with

(6.4) $$a_i^j = \begin{pmatrix} 1+\bar{u}' & -\xi\theta' & \eta\theta' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

(6.5) $$G^i = b_j^i \hat{e}^j$$

with

(6.6) $$b_j^i = \begin{pmatrix} 1-\bar{u}' & 0 & 0 \\ \xi\theta' & 1 & 0 \\ -\eta\theta' & 0 & 1 \end{pmatrix}$$

and

(6.7) $$\sqrt{G} = 1+\bar{u}' \ .$$

According to (6.1) with (6.4),(6.5) and (6.7), the Lagrange components in the rod cross section are:

(6.8)
$$\hat{T}^{11} = \hat{\tau}^{11} = \tau^{11}(1+2\bar{u}')$$
$$\hat{T}^{12} = \hat{\tau}^{12} = -\tau^{11}\xi\theta' + \tau^{12}(1+\bar{u}')$$
$$\hat{T}^{13} = \hat{\tau}^{13} = \tau^{11}\eta\theta' + \tau^{13}(1+\bar{u}') \ .$$

The longitudinal force acting on the rod in the direction of the tangent to the deformed rod axis results from the surface integral across the cross section. /37

(6.9) $$N = \int_F \hat{\tau}^{11}\,dF = \int_F \hat{T}^{11}\,dF = EF\bar{u}' \ .$$

The torsion moment is formed by the torsion shear stresses.

$$(6.10) \qquad D=\int_F (\eta\hat{\hat{\tau}}^{11}-\xi\hat{\hat{\tau}}^{12})dF=\int_F (\eta\hat{\tau}^{11}-\xi\hat{\tau}^{12})dF$$

If one confuses the Euler and Kirchhoff stresses, one gets a coupling of longitudinal force and torsion.

$$(6.11) \qquad D=\int_F [(1+\bar{u}')(\eta\tau^{11}-\xi\tau^{12})+(\eta^2+\xi^2)\theta'\tau^{11}]dF.$$

As can be seen from (6.11), the second term under the integral yields the common expression with the polar moment of inertia.

$$(6.12) \qquad J_p\,\theta'\tau^{11}=J_p\,\theta'\frac{N}{F}=i_p^2 N\theta'.$$

This term from integration (5.11) disappears, however, if one takes into account the fact that the shear stresses τ^{12} and τ^{13} cannot even approximately be set equivalent to each other in the context of the precision required here. Rather, the inverse of (2.23) as given in (2.33) must apply.

$$(6.13) \qquad \tau^{ij}=b_k^i\, b_l^j\, \hat{\hat{\tau}}^{kl}$$

with

$$(6.14) \qquad \begin{aligned} \tau^{11}&=\hat{\hat{\tau}}^{11}(1-2\bar{u}') \\ \tau^{12}&=\hat{\hat{\tau}}^{11}\xi\theta'+\hat{\hat{\tau}}^{12}(1-\bar{u}') \\ \tau^{13}&=-\hat{\hat{\tau}}^{11}\eta\theta+\hat{\hat{\tau}}^{13}(1-\bar{u}'). \end{aligned}$$

If one inserts (6.14) in (6.11), the result remains as /38 shown in (6.10).

For physically obvious reasons, longitudinal force can not at all be coupled with torsion in the case of curvature-free cross sections. Since curvature-free cross sections do not curve, the

movements of a point in the cross section due to strain in the direction $\hat{e}_1$ and torsion in the plane $(\hat{e}_2, \hat{e}_3)$ are exactly perpendicular to each other in the context of a rod theory. Coupling is thus ruled out.

However, the case of curvable cross sections is different. The example chosen here is a thin-walled open cross section, with a contour midline that must be free of shear stresses because of the closed shear lines. For the curvature function one then gets:

$$\left(\frac{\partial \varphi_s}{\partial \eta}\right)_m = \xi_m \quad \text{and} \quad \left(\frac{\partial \varphi_s}{\partial \xi}\right)_m = -\eta_m \quad (6.15)$$

Since for a rod it must be that $\hat{\tau}^{22} = \hat{\tau}^{33} = \hat{\tau}^{23} = \hat{\tau}^{32} = 0$, the Lagrange stresses still result as

$$\hat{T}^{12} = \hat{\tau}^{12} \quad \text{and} \quad \hat{T}^{13} = \hat{\tau}^{13} \quad (6.16)$$

But stresses $\hat{\tau}^{12}$ and $\hat{\tau}^{13}$ in this case already contain the coupling with strain. If one takes into account in (4.12) that $\bar{u}' = N/EF$, the coupling can be included even in a case without pre-twisting.

$$\begin{aligned} \hat{\tau}^{12} &= \hat{\tau}^{12}_D + \hat{\tau}^{12}_w - 2\mu\theta' \frac{N}{EF}\xi_m \\ \hat{\tau}^{13} &= \hat{\tau}^{13}_D + \hat{\tau}^{13}_w + 2\mu\theta' \frac{N}{EF}\eta_m \end{aligned} \quad (6.17)$$

If one now applies the surface integral (6.10) to (6.17), one gets the sought-for effect of the longitudinal force on torsion.

The considerations here make it clear that rod-beam theories

can include non-linear effects only to a certain degree. If a still greater degree of precision is to be achieved, it cannot be in the form of a rod-beam theory. /39

The simpler rod-beam theory, which only takes linear terms of deformation into account and includes effects of a higher order if they are given by known load functions, requires the same precision in its differential geometry assumptions as does a non-linear theory. If this is not provided for, one gets varying results that cannot be brought into accord with each other.

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